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Short Time Conservation of Gibbsianness Under Local Stochastic Evolutions

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We prove that a Gibbs measure with a finite range interaction evolved under a general reversible local stochastic dynamics remains Gibbsian for a short interval of time. This generalizes previous results for Glauber dynamics.

KEY WORDS: Gibbs versus non-Gibbs; interacting particle systems; cluster expansion.

1. INTRODUCTION

The evolution of a Gibbs measure under a stochastic dynamics is studied in ref. 4 in the context of high-temperature Glauber dynamics. This dynamics, when started from a low temperature state, can be interpreted as "heating the system." The question is whether at any time $t > 0$ a reasonable Hamiltonian H_t can be associated to the measure at time t, i.e., such that informally $\mu_t = e^{-H_t}$. If we start from a low-temperature state, then it can happen that no reasonable H_t exists. Since high-temperature dynamics started from any initial measure converges exponentially fast to the unique stationary measure (which is a high-temperature Gibbs measure), it is slightly surprising that non-Gibbsian measures could arise in the course of such an evolution. The presence of transitions Gibbs to non-Gibbs in the course of stochastic evolutions also provides a new motivation for the study of non-Gibbsian measures and the search for a reasonable extension of the present Gibbsian formalism. All results in ref. 4 are obtained for

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Glauber dynamics (spin-flips) and the "single-site" character of this dynamics was strongly used.

It is therefore natural to study the time evolution of a Gibbs measure under more general stochastic dynamics, such as Kawasaki or a mixture of Glauber and Kawasaki dynamics. In ref. 13 it is proved that a product measure evolved under a dynamics sufficiently close to independent spin flip dynamics remains a Gibbs measure for all times. This result includes the case of a small Kawazaki perturbation of independent spin flips but does not include, e.g., the case of a (even infinite temperature) Kawazaki dynamics.

In this paper, we concentrate on the short time behavior of a Gibbs measure under a general reversible local stochastic dynamics. We prove that for a short interval of time (depending on the range of the dynamics and of the range of the initial measure) a Gibbs measure with finite range interaction remains Gibbs, thus extending Theorem 4.1 of ref. 4. The intuition behind this result is simply that for short times "almost nothing changes," i.e., a typical trajectory consists of a "sea" of lattice sites where the configuration remains constantly equal to the initial one (non-active sites) and "isolated islands" of sites where the configuration changed (active sites).

The technical tool to formalize this intuitive picture is a generalization of a space time cluster expansion used in the proof of Theorem 4.1 of ref. 4. The polymer weights in this expansion are controlled for small times by the overwhelming small probability that activity occurred inside a polymer, where activity means that at every site of the polymer the governing Poisson clock rang at least once. The factorization property is obtained naturally in the case of non-interacting dynamics which can be viewed as generated by independent Poisson processes. In the interacting case, a Girsanov formula is used to go back to the non-interacting case.

Intuitively, it is not entirely surprising that conservation of the Gibbs property for short times is a rather robust statement, only dependent on locality. However, we expect that the presence of transitions Gibbs-non-Gibbs is very sensitive to the type of dynamics considered and in particular to the presence of conserved quantities.

Our paper is organized as follows. In Section 2 we define Gibbs measures and introduce the local stochastic dynamics which we use. In Section 3 we state our results and Section 4 is devoted to proofs. For the sake of notational simplicity, we do the complete proof for the easiest (non-Glauber) non-interacting dynamics: the simple symmetric exclusion process (i.e., non-interacting Kawasaki dynamics) starting from a Gibbs measure with nearest neighbor interaction. We then obtain the same result for general reversible local non-interacting dynamics starting from a finite range Gibbs measure as a rather straightforward generalization. To include the interaction of the dynamics, we use a Girsanov's formula and treat the extra factors due to the dynamics as an additional "interaction on trajectories."

2. NOTATIONS AND DEFINITIONS

2.1. Configuration Space

We consider spin systems on the lattice $\mathbb{Z}^d$. The configuration is given by a map $\sigma: \mathbb{Z}^d \to \{0, 1\}$ where we interpret $\sigma(x) = 1$, resp. $\sigma(x) = 0$, as the presence, resp. the absence, of a particle at site x. The set of all configurations is denoted by $\Omega = \{0, 1\}^{\mathbb{Z}^d}$. With the product topology, this is a compact metric space. $\mathscr{S}$ is denoted to be the set of all finite subsets of $\mathbb{Z}^d$ and for $A \subset \mathbb{Z}^d$, $\mathscr{F}_A$ denotes the σ-field generated by $\{\sigma(x): x \in A\}$; we write it $\mathscr{F}$ when $A = \mathbb{Z}^d$. For $\sigma, \xi \in \Omega$, we denote $\sigma_\Lambda \xi_{\Lambda^c}$ the configuration defined by

$$(\sigma_\Lambda \xi_{\Lambda^c})(x) = \begin{cases} \sigma(x) & \text{if} \quad x \in \Lambda. \\ \xi(x) & \text{if} \quad x \notin \Lambda. \end{cases}$$

The distance between $x = (x_i)_{i=1 \cdots d}$ and $y = (y_i)_{i=1 \cdots d}$ is $|x - y| = \sum_{i=1}^d |x_i - y_i|$, and if $|x - y| = 1$, we write $\langle xy \rangle$ meaning that $\langle xy \rangle$ is a nearest-neighbor bond. For $\Lambda \in \mathscr{S}$, the set of all nearest-neighbor bonds in Λ is denoted by

$$B(\Lambda) = \{\langle xy \rangle: |x - y| = 1, x, y \in \Lambda\}.$$

Between bonds, we define a distance d; for $b = \langle xy \rangle$, $b' = \langle x'y' \rangle$,

$$d(b, b') = \min\{|x_1 - x_2|, x_1 \in \{x, y\}, x_2 \in \{x', y'\}\}.$$

A function $f: \Omega \to \mathbb{R}$ is *local* if $\exists A \in \mathscr{S}$, $f \in \mathscr{F}_A$. The set of all local functions is denoted $\mathscr{L}$ and any continuous $f: \Omega \to \mathbb{R}$ is the uniform limit of elements in $\mathscr{L}$. $C(\Omega)$ denotes the set of all continuous functions. For $x \in \mathbb{Z}^d$, τ_x denotes the configuration shifted by x, i.e.,

$$(\tau_x \sigma)(y) = \sigma(x + y),$$

and similarly τ_x acts on functions via $(\tau_x f)(\sigma) = f(\tau_x \sigma)$, and on measures via $(\tau_x \mu)[f] = \mu[\tau_x f]$ for any local function.

2.2. Interactions, Gibbs Measures

An interaction is a map $\Phi\colon \mathcal{S} \times \Omega \to \mathbb{R}$ such that

1. $\Phi(A, \cdot) \in \mathcal{F}_A$, $\forall A \in \mathcal{S}$.
2. Φ is uniformly absolutely summable (UAS), i.e., for all $x \in \mathbb{Z}^d$,

$$\sum_{A \ni x} \sup_{\sigma \in \Omega} |\Phi(A, \sigma)| < \infty.$$

An interaction is translation-invariant if for all $A \in \mathcal{S}$, $\Phi(A+x, \sigma) = \Phi(A, \tau_x \sigma)$. In that case UAS is equivalent with

$$\|\Phi\|_1 = \sum_{A \ni 0} \sup_{\sigma \in \Omega} |\Phi(A, \sigma)| < \infty$$

and the set of all translation-invariant interactions forms a Banach space $\mathcal{B}_1$ with norm $\|\cdot\|_1$. An interaction Φ is called finite range if there exists $R > 0$ such that $\mathrm{diam}(A) > R$ implies that $\Phi(A, \cdot) = 0$. We denote $\mathcal{B}_{\text{f.r.}}$ the set of all translation invariant finite range interactions. Given $\Phi \in \mathcal{B}_1$, $\Lambda \in \mathcal{S}$, the Hamiltonian $H_\Lambda(\sigma \mid \xi)$ with boundary condition ξ is given by the absolutely convergent series

$$H_\Lambda(\sigma \mid \xi) = \sum_{A \cap \Lambda \neq \varnothing} \Phi(A, \sigma_\Lambda \xi_{\Lambda^c}),$$

also denoted $H^\xi_\Lambda(\sigma)$, and the Gibbs measure μ^ξ_Λ (at finite volume Λ and boundary condition ξ) is defined on $(\Omega_\Lambda, \mathcal{F}_\Lambda)$ by

$$\mu^\xi_\Lambda(A) = \frac{\sum_{\sigma \in \Omega_\Lambda} \mathbf{1}_A(\sigma)\, e^{-H^\xi_\Lambda(\sigma)}}{Z^\xi_\Lambda}$$

where

$$Z^\xi_\Lambda = \sum_{\sigma \in \Omega_\Lambda} e^{-H^\xi_\Lambda(\sigma)}$$

is the normalizing constant. A measure μ on $(\Omega, \mathcal{F})$ is a Gibbs measure with the interaction Φ, notation $\mu \in \mathcal{G}(\Phi)$, if and only if the finite-volume Gibbs measures μ^ξ_Λ for different ξ and Λ form a version of the conditional probabilities of μ, i.e., if

$$\mu^\xi_\Lambda(A) = \mu[A \mid \mathcal{F}_{\Lambda^c}](\xi) \;\; \mu\text{-a.s.}, \qquad \forall A \in \mathcal{F}. \tag{2.1}$$

A measure is called Gibbs if it is an element of $\mathcal{G} = \bigcup_{\Phi \in \mathcal{B}_1} \mathcal{G}(\Phi)$. (2.1) implies that every $\mu \in \mathcal{G}$ admits a continuous version of its conditional probabilities. Up to a non-nullness requirement, this condition is necessary and sufficient (see refs. 8, 15, and 5). An equivalent characterization for μ to be Gibbs (see ref. 3) is the following. For $x \in \mathbb{Z}^d$, let σ^x denote the configuration σ flipped at x, i.e.,

$$\sigma^x(y) = (1-\sigma(x))\,\delta_{x,y} + (1-\delta_{x,y})\,\sigma(y).$$

For μ a probability measure on $(\Omega, \mathcal{F})$, μ^x denotes the corresponding transformed measure:

$$\int f(\sigma)\,\mu^x(d\sigma) = \int f(\sigma^x)\,\mu(d\sigma).$$

The following relation between conditional probabilities and the Radon–Nikodym derivatives is obvious:

$$\mu[\sigma(x) \mid \mathcal{F}_{\mathbb{Z}^d \setminus x}](\xi) = \frac{1}{1 + \frac{d\mu^x}{d\mu}\,[\sigma_x \xi_{\mathbb{Z}^d \setminus x}]} \tag{2.2}$$

and therefore the one-site conditional probabilities are continuous if and only if the Radon–Nikodym derivatives $\frac{d\mu^x}{d\mu}$ are continuous. This gives

Lemma 2.3. A probability measure μ on Ω is a Gibbs measure if and only if the Radon–Nikodym derivative $\frac{d\mu^x}{d\mu}$ admits a continuous version for all $x \in \mathbb{Z}^d$.

Note that $\frac{d\mu^x}{d\mu} \in C(\Omega)$ automatically implies that $\frac{d\mu^x}{d\mu}$ is bounded away from zero and infinity since

$$\frac{d\mu^x}{d\mu}(\sigma) = \left(\frac{d\mu^x}{d\mu}(\sigma^x)\right)^{-1}.$$

2.3. Dynamics

Our dynamics are Feller processes generated by local rates. These processes have path-space measures $\mathbb{P}_\sigma$ concentrating on the space $D([0,t], \Omega)$ of càdlàg trajectories ω: $[0,t] \to \Omega$. We focus on three cases:

1. Kawasaki dynamics (exclusion with speed change).

2. Kawasaki+Glauber dynamics (exclusion with speed change plus births and deaths of particles).

3. General reversible local dynamics.

Of course, cases 1 and 2 are contained in case 3, but in the proof we will restrict to cases 1 and 2 and show afterwards how to generalize to case 3. This allows us to avoid setting up a labyrinth of unnecessary complicated notations. We now define the different types of dynamics more in detail.

2.3.1. Kawasaki Dynamics

The particles occupations are exchanged in configuration σ for nearest-neighbor bonds $\langle xy \rangle$ at rate $c(x, y, \sigma)$. More precisely, the process is defined by the generator L acting on $f \in \mathscr{L}$:

$$(Lf)(\sigma) = \sum_{\langle xy \rangle} c(x, y, \sigma)[f(\sigma^{xy}) - f(\sigma)], \tag{2.4}$$

where

$$\sigma^{xy}(z) = (1 - \delta_{z,x})(1 - \delta_{z,y})\, \sigma(z) + \delta_{x,z}\sigma(y) + \delta_{y,z}\sigma(x).$$

In words, σ^{xy} is the configuration obtained from σ by exchanging particle occupation numbers at site x and y. The special "non-interacting" case, where $c(x, y, \sigma) = 1$ for all $\langle xy \rangle$ and σ, corresponds to the simple symmetric exclusion process (SSE). We impose the following conditions on the rates:

1. Translation invariance: for all $x, y \in \mathbb{Z}^d$, $\sigma \in \Omega$: $c(x, y, \sigma) = c(0, y-x, \tau_x \sigma)$.

2. Strict positivity: $c(x, y, \sigma) > 0$, for all nearest neighbor bonds $\langle xy \rangle$, $\sigma \in \Omega$.

3. Locality: $c_{xy}: \sigma \mapsto c(x, y, \sigma) \in \mathscr{L}$ for all $x, y \in \mathbb{Z}^d$.

For ν a probability measure on $(\Omega, \mathscr{F})$, ν^{xy} is defined via its action on local functions f

$$\int f(\sigma)\, \nu^{xy}(d\sigma) = \int f(\sigma^{xy})\, \nu(d\sigma).$$

We then ask

4. Detailed balance for a Gibbs measure ν: there exists $\nu \in \mathscr{G}$ with $\Phi^\nu \in \mathscr{B}_{\text{f.r.}}$ such that

$$\frac{c(x, y, \sigma)}{c(x, y, \sigma^{xy})} = \frac{d\nu^{xy}}{d\nu}(\sigma) = \exp\left[\sum_{A \cap \{x, y\} \neq \varnothing} \Phi^\nu(A, \sigma) - \Phi^\nu(A, \sigma^{xy})\right] \tag{2.5}$$

for ν-almost every $\sigma \in \Omega$.

In ref. 11, the existence of a unique Feller process $\mathbb{P}_\sigma$ (starting from $\sigma \in \Omega$) with generator L is proved. We denote its semi-group by $(S(t))_{t \in \mathbb{R}^+}$:

$$\forall t \geqslant 0, \qquad \forall f \in \mathscr{L}, \qquad (S(t)\, f)(\sigma) = \mathbb{E}_\sigma[f(\sigma_t)]$$

and for μ a probability measure, we define $\mu S(t)$ via

$$\int f(\sigma)(\mu S(t))(d\sigma) = \int \mu(d\sigma)\, S(t)\, f(\sigma).$$

The fourth condition implies that the Gibbs measure ν is reversible for the process with generator L, i.e., when started from ν, the processes $\{\sigma_t: 0 \leqslant t \leqslant T\}$ and $\{\sigma_{T-t}: 0 \leqslant t \leqslant T\}$ are equal in distribution, or, equivalently, L and its semi-group $S(t) = \exp(tL)$ are self-adjoint operators on $\mathbb{L}^2(\nu)$.

In the case $c(x, y, \sigma) = 1$ (SSE), the reversible measures ν coincide with the Bernoulli product measures ν_ρ, $0 \leqslant \rho \leqslant 1$ corresponding to single-site interactions Φ^ν (homogeneous magnetic fields). In general, ν will not be unique since the dynamics has a conserved quantity.

2.3.2. *Kawasaki + Glauber*

In that case, the generator is given by

$$(Lf)(\sigma) = \sum_{x \in \mathbb{Z}^d} c(x, \sigma)[f(\sigma^x) - f(\sigma)] + \sum_{\langle xy \rangle} c(x, y, \sigma)[f(\sigma^{xy}) - f(\sigma)], \tag{2.6}$$

where the extra birth and death rates $c(x, \sigma)$ satisfy

1. Translation invariance: for all $x \in \mathbb{Z}^d$, $\sigma \in \Omega$: $c(x, \sigma) = c(0, \tau_x \sigma)$.
2. Strict positivity: $c(x, \sigma) > 0$, for all $x \in \mathbb{Z}^d$, $\sigma \in \Omega$.
3. Locality: $c_x: \sigma \to c(x, \sigma) \in \mathscr{L}$ for all $x \in \mathbb{Z}^d$.

4. Detailed balance: for a measure $\nu \in \mathcal{G}$ with $\Phi^\nu \in \mathcal{B}_{\text{f.r.}}$ of Section 2.3.1,

$$\frac{c(x,\sigma)}{c(x,\sigma^x)} = \frac{d\nu^x}{d\nu}(\sigma) = \exp\left[\sum_{A \ni x} \Phi^\nu(A,\sigma) - \Phi^\nu(A,\sigma^x)\right] \tag{2.7}$$

for ν-almost every $\sigma \in \Omega$.

The special case $c(x, y, \sigma) = c(x, \sigma) = 1$ corresponds to simple symmetric exclusion with independent births and deaths of particles. In that case, the Bernoulli measure $\nu_{\frac{1}{2}}$ is reversible.

2.3.3. General Reversible Local Dynamics

This section generalizes the two previous examples: instead of considering only spin-flip and spin-exchange we can allow more general transformations which change the configuration locally. More precisely, consider a set of transformations $\mathcal{T}_0$ such that every $T \in \mathcal{T}_0$ is a local bijection $T: \Omega \to \Omega$, i.e., there exists $\Lambda(T) \in \mathcal{S}$ with $(T(\sigma))(y) = \sigma(y)$ for all $y \notin \Lambda(T)$. To $T \in \mathcal{T}_0$, we associate the rate $c(T, \sigma)$ which is assumed to be a strictly positive function of σ. We then define the generator corresponding to the set $\mathcal{T}_0$ as

$$\forall \sigma \in \Omega, (Lf)(\sigma) = \sum_{x \in \mathbb{Z}^d} \sum_{T \in \mathcal{T}_x} c(x, T, \sigma)[f(T\sigma) - f(\sigma)] \tag{2.8}$$

where $\mathcal{T}_x = \{\tau_x \circ T_0 \circ \tau_{-x} : T_0 \in \mathcal{T}_0\}$ and for $T \in \mathcal{T}_x$, such that $T = \tau_x \circ T_0 \circ \tau_{-x}$, $c(x, T, \sigma) = c(T_0, \tau_{-x}\sigma)$. This definition ensures translation invariance of the dynamics. In words, this dynamics acts as follows: at each site x, we locally transform the configuration σ according to the transformation T_x, at rate $c(x, T_x, \sigma)$. The particular non-interacting case $c(x, T_x, \sigma) = 1$ corresponds to application of the transformation T_x at the event times of independent rate one Poisson processes. In that case, the Bernoulli measure $\nu_{\frac{1}{2}}$ is reversible. In the interacting case, we can impose the existence of a reversible $\nu \in \mathcal{G}$ with $\Phi^\nu \in \mathcal{B}_{\text{f.r.}}$, i.e., ν satisfies

$$\frac{d\nu T_x}{d\nu} = \frac{c(x, T_x, \sigma)}{c(x, T_x, T_x^{-1}\sigma)},$$

where νT_x is defined via

$$\int f(\sigma)\, \nu T_x(d\sigma) = \int f(T_x\sigma)\, \nu(d\sigma), \qquad \forall x \in \mathbb{Z}^d, \quad T_x \in \mathcal{T}_x.$$

In the examples of the two previous sections, the transformations $T \in \mathscr{T}_0$ are spin-flip at the origin $T\sigma = T_0\sigma = \sigma^0$, and spin-exchange of nearest neighbor bonds around the origin $T\sigma = T_{0e}\sigma = \sigma^{0e}$.

2.4. Poisson Representation of Non-Interacting Cases

In the non-interacting case (SSE, SSE+BD, $c(T,\sigma) = 1$), we have a simple representation of the process generated by a generator L^0 in terms of independent rate one Poisson processes. We describe this representation here in the three different cases.

Simple symmetric exclusion process (SSE):

$$L^0 f(\sigma) = \sum_{\langle xy \rangle} [f(\sigma^{xy}) - f(\sigma)]. \tag{2.9}$$

Given a collection of independent (rate one) Poisson processes indexed by nearest neighbor bonds, $\{N_t^{\langle xy \rangle} : t \geqslant 0, \langle xy \rangle \in B(\mathbb{Z}^d)\}$, a version of the process with generator L^0 is obtained by applying $\sigma \mapsto \sigma^{xy}$ at each event time of the Poisson process $N_t^{\langle xy \rangle}$.

SSE+birth and death (SSE+BD):

$$L^0 f(\sigma) = \sum_{\langle xy \rangle} [f(\sigma^{xy}) - f(\sigma)] + \sum_x [f(\sigma^x) - f(\sigma)]. \tag{2.10}$$

The collection of independent (rate one) Poisson processes $\{N_t^{\langle xy \rangle} : t \geqslant 0, \langle xy \rangle \in B(\mathbb{Z}^d)\} \cup \{N_t^x : t \geqslant 0, x \in \mathbb{Z}^d\}$ is now indexed by both bonds and sites. A version of the process with generator L^0 is obtained as follows: at the event times of $N_t^{\langle xy \rangle}$, apply $\sigma \mapsto \sigma^{xy}$, at the event times of N_t^x, apply $\sigma \mapsto \sigma^x$.

General case:

$$L^0 f(\sigma) = \sum_x \sum_{T \in \mathscr{T}_x} [f(T\sigma) - f(\sigma)]. \tag{2.11}$$

Consider the collection of independent (rate one) Poisson processes $\{N_t^T : t \geqslant 0, T \in \mathscr{T}_x, x \in \mathbb{Z}^d\}$. A version of this process is obtained by applying $\sigma \to T\sigma$ at the event times of N_t^T.

3. RESULT AND SKETCH OF PROOF

We consider a local dynamics with generator (2.8), and corresponding reversible Gibbs measure ν. We start the dynamics from a Gibbs measure μ.

To avoid trivialities, the interaction $\Phi^\mu \in \mathscr{B}_{\text{f.r.}}$ is chosen such that $\mathscr{G}(\Phi^\mu) \neq \mathscr{G}(\Phi^\nu)$, i.e., Φ^μ and Φ^ν are not physically equivalent. We then have

Theorem 3.1. There exists $t_0 = t_0(\Phi^\mu, \Phi^\nu) > 0$ such that for all $t \leqslant t_0$, $\mu_t = \mu S(t)$ is a Gibbs measure.

The rest of the paper is devoted to the proof of Theorem 3.1. The main steps are:

1. Non-interacting case: SSE.
2. Non-interacting case: SSE + BD.
3. Interacting case: Kawasaki + Glauber.
4. General case.

We consider $\Lambda \in \mathscr{S}$, $\sigma \in \Omega$, and abbreviate

$$\mathscr{H}_\Lambda(\sigma) = \sum_{A \subset \Lambda} [\Phi^\nu(A, \sigma) - \Phi^\mu(A, \sigma)]$$

and with a boundary condition $\xi \in \Omega$,

$$\mathscr{H}^\xi_\Lambda(\sigma) = \sum_{A \cap \Lambda \neq \varnothing} [\Phi^\nu(A, \sigma_\Lambda \xi_{\Lambda^c}) - \Phi^\mu(A, \sigma_\Lambda \xi_{\Lambda^c})].$$

We also denote $\mathbb{P}^\Lambda_\sigma$ for the path-space measure of the process in volume Λ (with free boundary conditions) and $\mathbb{E}^\Lambda_\sigma$ the corresponding expectation. For the Kawasaki dynamics, e.g., this means that in the process with law $\mathbb{P}^\Lambda_\sigma$ exchanges along the bond $\langle xy \rangle$ are performed at rate $c(x, y, \sigma)$ if this function depends only on the spins $\sigma(z)$, $z \in \Lambda$. For the other bonds $\langle xy \rangle$ inside Λ, the exchange rate is chosen such that $c(x, y, \sigma)$ $(x, y \in \Lambda)$ satisfies

$$\frac{c(x, y, \sigma)}{c(x, y, \sigma^{xy})} = \frac{d\nu^{xy}_\Lambda}{d\nu_\Lambda}(\sigma) = \exp\left[\sum_{A \cap \{x, y\} \neq \varnothing,\, A \subset \Lambda} \Phi^\nu(A, \sigma) - \Phi^\nu(A, \sigma^{xy})\right] \tag{3.2}$$

For this finite volume process, the free boundary condition Gibbs measure

$$\nu_\Lambda(\sigma_\Lambda) = \frac{1}{Z_\Lambda} \exp\left(-\sum_{A \subset \Lambda} \Phi^\nu(A, \sigma)\right)$$

is reversible.

The following lemma is proved in ref. 4.

Lemma 3.3. If, for $x \in \mathbb{Z}^d$, the sequence of functions

$$\Psi^x_{\Lambda,t} : \sigma \mapsto \frac{\mathbb{E}^{\Lambda}_{\sigma^x}[e^{\mathscr{H}_\Lambda(\sigma_t)}]}{\mathbb{E}^{\Lambda}_{\sigma}[e^{\mathscr{H}_\Lambda(\sigma_t)}]} \tag{3.4}$$

converges uniformly as $\Lambda \uparrow \mathbb{Z}^d$ to a continuous function Ψ^x_t, then the function $\Psi^x_t \cdot \frac{dv^x}{dv}$ is a continuous version of the Radon–Nikodym derivative $\frac{d\mu S(t)^x}{d\mu S(t)}$, and the measure $\mu S(t)$ is Gibbs.

The strategy to prove that $\Psi^x_{\Lambda,t}$ converges uniformly is to obtain a convergent cluster expansion of

$$\ln \mathbb{E}^{\Lambda}_{\sigma}[e^{\mathscr{H}_\Lambda(\sigma_t)-\mathscr{H}_\Lambda(\sigma_0)}] = \sum_{\Gamma \subset \Lambda} a(\Gamma)\, w^t_\sigma(\Gamma).$$

where $a(\Gamma)$ are combinatorial (σ-independent) factors and $w^t_\sigma(\Gamma)$ are cluster weights. As long as t is sufficiently small, the configuration σ_t can be seen as a sea of the initial configuration $\sigma_0 = \sigma$ and isolated islands where something changed. The cluster weights $w^t_\sigma(\Gamma)$ are then controlled for small $t > 0$ via the Kotecký–Preiss criterion,[7] uniformly in σ and will give us uniform absolute convergence of the series

$$\sum_{\Gamma \ni x} a(\Gamma)\, w^t_\sigma(\Gamma)$$

which clearly implies uniform convergence of the quotient in (3.4).

Remark. In Lemma 3.2, the reversibility of the dynamics is used. This assumption can be replaced by the condition that the dynamics admits a *stationary* Gibbs measure v with finite range interaction. In that case, in (3.4) we have to replace the expectations $\mathbb{E}_\sigma$ by expectations in the time-reversed process with semigroup $S^*(t)$ (the adjoint of $S(t)$ in $L^2(v)$).

4. PROOF OF THE THEOREM

4.1. Non-Interacting Case SSE

4.1.1. Φ^μ Nearest Neighbor

By Lemma 3.3, it suffices to prove the uniform convergence of

$$\frac{\mathbb{E}^{\Lambda}_{\sigma^x}[e^{\mathscr{H}_\Lambda(\sigma_t)-\mathscr{H}_\Lambda(\sigma_0)}]}{\mathbb{E}^{\Lambda}_{\sigma}[e^{\mathscr{H}_\Lambda(\sigma_t)-\mathscr{H}_\Lambda(\sigma_0)}]}$$

for t small enough. We remind the notation $\mathscr{H}_\Lambda(\sigma) = \sum_{A \subset \Lambda} [\Phi^\nu(A, \sigma) - \Phi^\mu(A, \sigma)]$ which in this case ($\Phi^\nu = 0$) reduces to $\mathscr{H}_\Lambda(\sigma) = -\sum_{A \subset \Lambda} [\Phi^\mu(A, \sigma)]$. For a given realization ω of the Poisson process $\{N_b^s : 0 \leqslant s \leqslant t, b \in B(\Lambda)\}$, we define the set of active bonds by

$$\mathscr{A}(\omega) = \{b \in B(\Lambda) : \exists b' \in B(\Lambda) \text{ s.t. } d(b, b') \leqslant 1 \text{ and } N_b^t + N_{b'}^t > 0\}. \tag{4.1}$$

Next we decompose $\mathscr{A}(\omega)$ into disjoint maximally connected components

$$\mathscr{A}(\omega) = \bigcup_{i = 1 \cdots n} \gamma_i(\omega).$$

We denote by $\partial\gamma$ the (nearest neighbor) inner boundary of a connected set of bonds γ and with these notations, we obtain

$$\mathbb{E}_\sigma^\Lambda[\exp(\mathscr{H}_\Lambda(\sigma_t) - \mathscr{H}_\Lambda(\sigma))] = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1, \ldots, \gamma_n)_c \subset \Lambda} \left\{ \prod_{i=1}^{n} w_\sigma^t(\gamma_i) \right\} e^{-t\,|B(\Lambda)|} \tag{4.2}$$

where the sum $\sum_{(\gamma_1, \ldots, \gamma_n)_c \subset \Lambda}$ is over all the *compatible* (disjoint) maximal components γ_i (called polymers in ref. 7) of Λ. With a slight abuse of notation we use the symbol γ both for a set of bonds and for the set of sites spanned by the bonds, i.e., the set $\{x \in \mathbb{Z}^d : \exists b \in \gamma \text{ with } b = \langle xy \rangle\}$. The weights are given by

$$w_\sigma^t(\gamma) = \mathbb{E}_\sigma^\Lambda[\exp(\mathscr{H}_\gamma^\sigma(\sigma_t) - \mathscr{H}_\gamma^\sigma(\sigma))\, F_\gamma(\omega)]\, e^{t\,|\gamma|} \tag{4.3}$$

where F_γ is a function of the realization ω of the Poisson processes, defined as follows:

$$\begin{aligned} F_\gamma(\omega) &= I\{\gamma \text{ is a maximally connected component of } \mathscr{A}(\omega)\} \\ &= \left(\prod_{b \in \partial\gamma} \mathbf{1}_{N_b^t = 0} \right) I\{\forall b \in \gamma, \exists b' \in \gamma, d(b, b') \leqslant 1, N_b^t + N_{b'}^t > 0\}. \end{aligned}$$

The factor $e^{t\,|\gamma|}$ arises from the probability

$$\mathbb{P}_\sigma^\Lambda \left[N_b^t = 0, \forall b \in B(\Lambda) \Big\backslash \bigcup_{i=1}^{n} \gamma_i \right] = \exp\left\{ -t \left[|B(\Lambda)| - \sum_{i=1}^{n} |\gamma_i| \right] \right\}.$$

The factorization of the weights in (4.2) follows from the fact that the function

$$\Psi_\gamma(\omega) = \exp(\mathscr{H}_\gamma^\sigma(\sigma_t) - \mathscr{H}_\gamma^\sigma(\sigma))\, F_\gamma(\omega) \tag{4.4}$$

is a function of the Poisson processes $\{N_b^s\colon 0 \leqslant s \leqslant t, b \in \gamma\}$. Therefore, by the independence of these Poisson processes, for $\gamma \cap \gamma' = \emptyset$, Ψ_γ and $\Psi_{\gamma'}$ are independent.

In order to apply the Kotecký–Preiss criterion[7] and to write down a convergent (uniformly in σ) expansion of the logarithm of the series in (4.2) for t small enough, it suffices to prove that the weights satisfy the bound

$$|w_\sigma^t(\gamma)| \leqslant e^{-c(t)\,|\gamma|} \tag{4.5}$$

where $c(t) \longrightarrow +\infty$ as $t \downarrow 0$ and is independent of σ. Indeed, by definition of the polymers γ, for $b \in B(\Lambda)$ one easily obtains the bound

$$|\{\gamma\colon \gamma \ni b, |\gamma| = n\}| \leqslant \exp(an) \tag{4.6}$$

where $a > 0$ is a constant depending only on the dimension. Therefore, the estimate (4.5) implies the the Kotecký Preiss criterion for t small enough. To obtain this bound (4.5), we use the following estimates

$$\sup_{\sigma, \xi, \eta} [\exp(\mathscr{H}_\gamma^\sigma(\xi) - \mathscr{H}_\gamma^\sigma(\eta))] \leqslant e^{|\gamma|\, C(\Phi^\mu)} \tag{4.7}$$

(where we can choose, e.g., $C(\Phi^\mu) = 2\sum_{A \ni 0} \|\Phi_A^\mu\|_\infty$) and the fact that $\exists c > 0$, $\alpha(d) > 0$, $\epsilon(d) \in\,]0, 1[$ such that

$$\mathbb{P}_\sigma^\Lambda[\forall b \in \gamma, \exists b' \in \gamma, d(b, b') \leqslant 1, N_b^t + N_{b'}^t > 0] \leqslant C(1 - e^{-\alpha(d)\, t})^{\epsilon(d)\, |\gamma|}. \tag{4.8}$$

To see this, first take $d = 1$; the event is then simply that of any two nearest neighbor bonds at least one had a Poisson event, i.e., we can choose $\alpha(1) = 2$, $\epsilon(1) = \frac{1}{2}$. For general d, any cube of size 2 contained in γ must have at least one Poisson event.

Combining (4.3), (4.7) and (4.8), we obtain estimate (4.5) if

$$C(1 - e^{-\alpha(d)\, t})^{\epsilon(d)}\, e^t e^{C(\Phi^\mu)} < 1 \tag{4.9}$$

which is realized as soon as t is small enough, i.e., $0 \leqslant t \leqslant t_0$ with $t_0 = t_0(\Phi^\mu)$. For such a t, we can write

$$\ln \mathbb{E}_\sigma^\Lambda[\exp(\mathscr{H}_\Lambda(\sigma_t) - \mathscr{H}_\Lambda(\sigma_0))] = \sum_{\Gamma \text{ m.i.} \subset \Lambda} a(\Gamma)\, w_\sigma^t(\Gamma)$$

where the sum over Γ runs over all clusters, i.e., multi-indices of compatible contours γ. The cluster weights $w_\sigma^t(\Gamma)$ and $w_{\sigma^x}^t(\Gamma)$ differ only for clusters Γ containing x. Moreover, since the estimate in (4.5) is uniform in σ, we have

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \sum_{\Lambda \ni x, \Gamma \cap \Lambda^c \neq 0} a(\Gamma) \sup_{\sigma} |w^t_\sigma(\Gamma)| = 0.$$

Therefore, writing

$$\varphi^x_{\Lambda,t}(\sigma) = \frac{\mathbb{E}^\Lambda_{\sigma^x}[e^{\mathscr{H}_\Lambda(\sigma_t) - \mathscr{H}_\Lambda(\sigma_0)}]}{\mathbb{E}^\Lambda_\sigma[e^{\mathscr{H}_\Lambda(\sigma_t) - \mathscr{H}_\Lambda(\sigma_0)}]} = \exp\left\{ \sum_{\Gamma \in \Lambda, \Gamma \ni x} a(\Gamma)[w^t_{\sigma^x}(\Gamma) - w^t_\sigma(\Gamma)] \right\}, \tag{4.10}$$

we conclude uniform convergence of $\varphi^x_{\Lambda,t}(\sigma)$ as $\Lambda \uparrow \mathbb{Z}^d$ for $t \leqslant t_0$, and hence the same holds for $\Psi^x_{\Lambda,t}$.

4.1.2. Φ^μ Finite Range

In this case, we redefine the set of active bonds

$$\mathscr{A}(\omega) = \{b \in b(\Lambda) : \exists b', d(b, b') \leqslant R_\mu \text{ and } N^t_{b'}(\omega) > 0\}$$

where R_μ is the range of the interaction of the starting Gibbs measure. We then decompose

$$\mathscr{A} = \bigcup_{i=1}^{n} \gamma_i$$

into maximally (nearest neighbor) connected contours and define the Rth inner boundary, resp. interior, of γ to be $\partial_R \gamma = \{x \in \gamma, \exists y \notin \gamma, |x - y| \leqslant R\}$, respectively $\gamma^\circ_R = \gamma \setminus \partial_R \gamma$. With these notations, we still have

$$\mathbb{E}^\Lambda_\sigma[\exp(\mathscr{H}_\Lambda(\sigma_t) - \mathscr{H}_\Lambda(\sigma))] = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)_c \subset \Lambda} \left\{ \prod_{i=1}^{n} w^t_\sigma(\gamma_i) \right\} e^{-t|B(\Lambda)|},$$

where

$$w^t_\sigma(\gamma) = \mathbb{E}^\Lambda_\sigma[\exp(\mathscr{H}^\sigma_\gamma(\sigma_t) - \mathscr{H}^\sigma_\gamma(\sigma))\, F_\gamma(\omega)]\, e^{t|\gamma|}$$

but

$$F_\gamma(\omega) = I\{\gamma \text{ is a maximally connected component of } \mathscr{A}(\omega)\}$$

is now written as

$$\left(\prod_{b \in \partial_R \gamma} \mathbf{1}_{N^t_b = 0} \right) G_{\gamma^\circ_R}(\omega)$$

with

$$G_{\gamma_R^\circ}(\omega) = I\{\forall b \in \gamma_R^\circ, \exists b' \in \gamma_R^\circ, d(b, b') \leqslant R, N_b^t + N_{b'}^t > 0\}.$$

Here we have the estimate

$$\mathbb{E}_\sigma^\Lambda[F_\gamma(\omega)] \leqslant (1 - e^{-t(R+1)^d})^{\frac{|\gamma_R^\circ|}{(R+1)^d}} \tag{4.11}$$

$$\leqslant ((1 - e^{-t(R+1)})^{\frac{1}{(R+1)^d}})^{|\gamma|} C_R. \tag{4.12}$$

From this estimate we obtain again like in (4.5)

$$|w_\sigma^t(\gamma)| \leqslant e^{-\alpha(R,t)\,|\gamma|}$$

where $\alpha(R, t) \to \infty$ as $t \downarrow 0$.

4.2. Non-Interacting Case SSE + BD

We consider the general case with Φ^μ finite range. We again need to redefine the active bonds. Given a trajectory ω of the process we call a site x active if there exists a bond $b \in B(\Lambda)$ such that $d(x, b) \leqslant R_\mu$ and $N_b^t(\omega) > 0$ or there exists a site $y \in \Lambda$ such that $d(x, y) \leqslant R_\mu$ and $N_y^t > 0$. We denote again $\mathscr{A}(\omega) = \{x \in \Lambda, x \text{ is active}\}$ and decompose it $\mathscr{A}(\omega) = \bigcup_{i=1}^n \gamma_i$, where γ_i are the mutually disjoint maximally connected components of $\mathscr{A}$. To set up a similar expansion, we introduce the following notation: for $A \in \mathscr{S}$, denote

$$\xi(A) = |\{\langle xy \rangle : x \in A, y \in A\}| + |A|. \tag{4.13}$$

With these notations, we write:

$$\mathbb{E}_\sigma^\Lambda[\exp(\mathscr{H}_\Lambda(\sigma_t) - \mathscr{H}_\Lambda(\sigma))] = e^{-t\xi(\Lambda)} \sum_{n=1}^\infty \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)_c \subset \Lambda} \left\{ \prod_{i=1}^n w_\sigma^t(\gamma_i) \right\},$$

where

$$w_\sigma^t(\gamma) = \mathbb{E}_\sigma^\Lambda[\exp(\mathscr{H}_\gamma^\sigma(\sigma_t) - \mathscr{H}_\gamma^\sigma(\sigma))\, F_\gamma(\omega)]\, e^{t\xi(\gamma)} \tag{4.14}$$

with

$$F_\gamma(\omega) = I\{\gamma \text{ is a maximally connected component of } \mathscr{A}(\omega)\}.$$

It is then easily verified that

$$\mathbb{E}_\sigma^\Lambda[F_\gamma(\omega)] \leqslant e^{-\alpha(R,t)\,|\gamma|}$$

where $\alpha(R, t) \to \infty$ as $t \downarrow 0$. Here, since $|\xi(\gamma)| \leqslant C\,|\gamma|$, we obtain

$$|w_\sigma^t(\gamma)| \leqslant e^{-\alpha'(R,t)\,|\gamma|}$$

where $\alpha'(R, t) \to \infty$ as $t \downarrow 0$.

4.3. General Non-Interacting Case

We consider general reversible local dynamics as introduced in (2.11) and start from a Gibbs measure for a general finite range interaction $\Phi^\mu \in \mathscr{B}_{\text{f.r.}}$ with range R_μ. The range R_d of the dynamics generated by the transformations in $\mathscr{T}_0$ is defined as the radius of the minimal ball $B(0, R_d)$ with center 0 such that for all $T \in \mathscr{T}_0$, $\Lambda(T) \subset B(0, R_d)$. We define $R = \max\{R_d, R(\Phi^\mu)\}$ and introduce the set of active sites for a trajectory $\omega \in D([0, t], \Omega_\Lambda)$:

$$\mathscr{A}(\omega) = \{x \in \Lambda : \exists y \in \Lambda,\, d(x, y) \leqslant R \text{ and } N_t^T > 0$$
$$\text{for some } T \in \textstyle\bigcup_z \mathscr{T}_z \text{ with } \Lambda(T) \ni y\}.$$

The same expansion as in Section 4.2. now applies after redefining

$$\xi(A) = |\{T \colon \Lambda(T) \subset A\}|.$$

4.4. General Case

We consider general local dynamics as introduced in Section 2.3.3, with a Gibbs measure ν for a finite range interaction Φ^ν as reversible invariant measure. We use a Girsanov formula (see ref. 1 or ref. 12) to go back to case 4.3. Denote $\mathbb{P}_\sigma^\Lambda$ for the path-space measure on $D([0, t], \Omega_\Lambda)$ of the interacting process in volume Λ, and $\mathbb{P}_{\sigma,0}^\Lambda$ for the path-space measure of the non-interacting case. We have

$$\frac{d\mathbb{P}_\sigma^\Lambda}{d\mathbb{P}_{\sigma,0}^\Lambda}(\omega) = \exp\left\{\sum_{x \in \Lambda}\sum_{T \in \mathscr{T}_x}\int_0^t \log c(x, T, \omega_s)\, dN_s^T + \int_0^t (c(x, T, \omega_s) - 1)\, ds\right\} \tag{4.15}$$

and hence

$$\mathbb{E}_{\sigma}^{\Lambda}[\exp\{\mathscr{H}_{\Lambda}(\sigma_t)\}]=\mathbb{E}_{\sigma,0}^{\Lambda}\left[\exp\left\{\mathscr{H}_{\Lambda}(\sigma_t)+\sum_{x\in\Lambda}\sum_{T\in\mathscr{T}_x}\left(\int_0^t\log c(x,T,\omega_s)\,dN_s^T\right.\right.\right.$$
$$\left.\left.\left.-\int_0^t(c(x,T,\omega_s)-1)\,ds\right)\right\}\right] \tag{4.16}$$

This can be written in the form $\mathbb{E}_{\sigma}^{\Lambda}(\exp\{\mathscr{U}_{\Lambda}^{t}\})$ where $\mathscr{U}$ is defined on trajectories $\omega\in D([0,t],\Omega_{\Lambda})$ by

$$\mathscr{U}_{\Lambda}^{t}(\omega)=\mathscr{H}_{\Lambda}(\omega(t))+\sum_{x\in\Lambda}\sum_{T\in\mathscr{T}_x}\left(\int_0^t\log c(x,T,\omega_s)\,dN_s^T\right.$$
$$\left.-\int_0^t(c(x,T,\omega_s)-1)\,ds\right). \tag{4.17}$$

We can now expand in a similar way the logarithm of the expectation

$$\mathbb{E}_{\sigma,0}^{\Lambda}(\exp(\mathscr{U}_{\Lambda}^{t}(\omega)-\mathscr{U}_{\Lambda}^{t}(\bar\sigma)) \tag{4.18}$$

where $\bar\sigma$ denotes the trajectory constantly equal to the initial configuration σ. In order to obtain factorization of the polymer weights, first introduce a new range related to the region affected by the transformations T, to the finite range potential Φ^{μ}, and to the range R' of the rates $c(x,T,\cdot)$ and define

$$R=\max\{R_{\mu},R_d,R'\}.$$

Using this R we define the active sites $\mathscr{A}(\omega)$ as in Section 4.3 and decompose them into maximally connected components γ_i. The only additional problem in the control of the polymer weights are the additional Girsanov factors, i.e., the polymer weights are given by the same expression as in (4.14), with $\mathscr{H}_{\gamma}(\sigma_t)$ replaced with $\mathscr{U}_{\gamma}^{t}(\omega)$ and $\mathscr{H}_{\gamma}(\sigma)$ replaced with $\mathscr{U}_{\Lambda}^{t}(\bar\sigma)$. To control the Girsanov factors in these weights, use

$$\mathbb{E}_{\sigma,0}^{\gamma}\left[\exp\left\{\sum_{x\in\gamma}\sum_{T\in\mathscr{T}_x}\int_0^t\log c(x,T,\omega_s)\,dN_s^T-\int_0^t(c(x,T,\omega_s)-1)\,ds\right\}\right]\leqslant e^{\alpha|\gamma|t}$$

where $0<\alpha<\infty$. This estimate is an immediate consequence of the fact that under the measure $\mathbb{P}_{\sigma,0}^{\gamma}$, $\{N^T: T\in\mathscr{T}_x, x\in\gamma\}$ are independent rate one Poisson processes.

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